Second Order Lax Pairs of Nonlinear Partial Differential Equations with Schwarzian Forms

Sen-yue Lou[a,b,c], Xiao-yan Tang[a,b], Qing-Ping Liu[b,d], and T. Fukuyama[e,f]

[a] Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, P. R. China
[b] Abdus Salam International Centre for Theoretical Physics, Trieste, Italy
[c] Department of Physics, Ningbo University, Ningbo 315211, P. R. China
[d] Beijing Graduate School, China University of Mining and Technology, Beijing 100083, P. R. China
[e] Department of Physics, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan
[f] Department of Physics, University of Maryland, College Park, MD20742, U.S.A.

Reprint requests to Dr. S.-y. L.; E-mail: sylou@mail.sjtu.edu.cn

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In this paper we study the possible second order Lax operators for all possible (1+1)-dimensional models with Schwarzian forms. If the Schwarzian form of a (1+1)-dimensional model can be expressed by two known conformal invariants (invariant under the Möbius transformation), the model has a second order lax pair. The explicit Lax pairs for some (1+1)-dimensional are given. The conclusions are also extended to some (2+1)-dimensional equations.

Key words: Lax Pairs; Schwarzian Forms; Möbius Transformation; Conformal Invariants.

1. Introduction

In the study of a nonlinear mathematical physics system, the integrablility condition of a pair of linear problems can offer a nonlinear system. Some types of special exact solutions of the nonlinear system can be obtained by means of a pair of linear problems. The pair of linear equations is called a Lax pair of the nonlinear system and the nonlinear system is called Lax integrable or IST (inverse scattering transformation) integrable if the Lax pair possesses a nontrivial spectral parameter. Usually, an IST integrable model also has many other interesting properties such as the existence of infinitely many conservation laws and infinitely many symmetries, multi-soliton solutions, bilinear forms, Schwarzian forms, multi-Hamiltonian structures, Painlevé property, and so on.

To our knowledge, almost all the known IST integrable (1+1)- and (2+1)-dimensional models can be transformed to some types of invariant forms which are invariant under the Möbius transformation. We call these types of invariant forms the Schwarzian forms of the original models because they are usually expressed by the Schwarzian derivatives. The truncated Painlevé expansion approach [1, 2] may be one of the best ways to find the Schwarzian forms of the original integrable models.

Our recent studies indicate that the existence of the Schwarzian forms plays an important role in the study of integrable systems. The conformal invariance (invariant under the Möbius transformation) of the well-known Schwarzian Korteweg de-Vries (SKdV) equation is related to the infinitely many symmetries of the usual KdV equation [3]. The conformal invariant related flow equation of the SKdV is linked with some types of (1+1)-dimensional and (2+1)-dimensional sinh-Gordon (ShG) equations and Mikhailov-Dodd-Bullough (MDB) equations [4]. In addition, by means of the Schwarzian forms of many known integrable models one can discover some other integrable properties like the Bäcklund transformations and Lax pairs [1]. In [5], one of the present authors (Lou) proposed that starting from a conformal invariant form may be one of the effective ways to find integrable models particularly in higher dimensions. Some types of quite general Schwarzian equations are Painlevé integrable. In [6], Conte's conformal invariant Painlevé analysis [7] has been extended to obtain high dimensional Painlevé integrable Schwarzian equations systematically, and some types of physically important high dimensional nonintegrable models can be solved approximately via some high dimensional Painlevé integrable Schwarzian equations [8].

Now an important question is what kind of Schwarzian equations are related to some Lax integrable models? To answer this question generally in arbitrary dimensions is quite difficult. So the models discussed in this paper are mainly focused on (1+1)-dimensional cases.

In the next section, we prove that for any (1+1)-dimensional Schwarzian model (which is described by two conformal invariants) there is a second order Lax pair. Various concrete physically significant examples are listed in Section 3. Section 4 is devoted to discussing some special extensions in high dimensions. The possibility to introduce some nontrivial spectral parameters in Lax pairs is discussed in Section 5. The last section is a short summary and discussion.

2. A Second Order (1+1)-dimensional Lax Pair Linked with an Arbitrary Schwarzian Form

In (1+1)-dimensions, the known independent conformal invariants are

$$p_1 \equiv \frac{\phi_t}{\phi_x},\ p_2 \equiv \{\phi;\ x\} \equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}, \quad (1)$$

where ϕ is a function of $\{x,\ t\}$, the subscripts are usual derivatives while $\{\phi;\ x\}$ is the Schwarzian derivative. As in [5, 6, 8], we say a quantity is a conformal invariant if it is invariant under the Möbius transformation

$$\phi \to \frac{a\phi + b}{c\phi + d},\ ad \neq bc. \quad (2)$$

From (1) we can write down a general (1+1)-dimensional conformal invariant Schwarzian equation

$$F(x, t, p_i, p_{ix}, p_{it}, p_{ixx}, \ldots (i = 1, 2)) \equiv F(p_1,\ p_2) = 0, \quad (3)$$

where F is an arbitrary function of $x,\ t,\ p_i$ and any order of derivatives and even integrations of p_i with respect to x and t. According to the idea in [5], (3) (or many of (3)) may be integrable. It is quite easy to check the Painlevé integrability of (3) by using the method given in [5, 6], when F is a polynomial function of p_i and the derivatives of p_i. But it is rather hard for a quite general form of the function F. Fortunately, we can find its relevant forms with Lax pair(s). To realize this idea, we consider the second order Lax pair

$$\begin{cases} \psi_{xx} = u\psi_x + v\psi, \\ \psi_t = u_1\psi_x + v_1\psi, \end{cases} \quad (4)$$

where $u,\ u_1,\ v$, and v_1 are undetermined functions. To link the Lax pair (4) with the Schwarzian equation (3), we suppose that ψ_1, ψ_2 are two solutions of (4) and ϕ of (3) is linked to ψ_1 and ψ_2 by

$$\phi = \frac{\psi_1}{\psi_2}. \quad (5)$$

Now by substituting (5) with (4) into (3) directly, we know that, if the functions $u,\ v$ and u_1 are linked by

$$F(P_1,\ P_2) = 0 \quad (6)$$

with

$$P_1 = u_1,\ P_2 = u_x - \frac{1}{2}u^2 - 2v, \quad (7)$$

then the corresponding nonlinear equation system for the fields $u,\ v,\ u_1$ and v_1 has a Lax pair (4), and the fields $u,\ v,\ u_1$ and v_1 are linked to the field ϕ by the non-auto-Bäcklund transformation

$$p_i = P_i, (i = 1,\ 2). \quad (8)$$

Finally, the evolution equation system is obtained straightforwardly by calculating the compatibility condition of (4),

$$\psi_{xxt} = \psi_{txx}. \quad (9)$$

The result reads

$$v_t = v_{1xx} + 2vu_{1x} + u_1v_x - uv_{1x}, \quad (10)$$

$$u_t = u_{1xx} + 2v_{1x} + (uu_1)_x \quad (11)$$

with the constraint (6). In (6), (10) and (11) one of the four functions $u,\ u_1,\ v$, and v_1 remains free. For simplicity we take

$$u = 0,\ v_1 = -\frac{1}{2}u_{1x}. \quad (12)$$

Due to the simplification (12), the final evolution equation related to the Schwarzian form (3) has the form of

$$v_t = -\frac{1}{2}u_{1xxx} + 2vu_{1x} + u_1v_x \tag{13}$$

with (6) for $u = 0$. Simultaneously, the Lax pair is simplified to

$$\begin{cases} L\psi \equiv (\partial_x^2 - v)\psi = 0, \\ \psi_t = M\psi \equiv (u_1\partial_x - \frac{1}{2}u_{1x})\psi. \end{cases} \tag{14}$$

What should be emphasized again is that the Lax operator given in (14) is only a second order operator.

For the sake of showing the results more concretely, we discuss some special physically significant models in the following section.

3. Special Examples

From suitable selections of $F \equiv F(p_1,\ p_2)$ in (3), various interesting examples are obtained, based on the general theory given in the last section.

Example 1. KdV equation.

The Schwarzian KdV equation has the simple form

$$F_{\text{KdV}}(p_i) = p_1 + p_2 = 0. \tag{15}$$

According to (6) with $u = 0$, we know that the relation between the functions v and u_1 is simply given by

$$v = \frac{1}{2}u_1. \tag{16}$$

Substituting (16) into (14), we reobtain the well known Lax pair

$$\begin{cases} \psi_{xx} - \frac{1}{2}u_1\psi = 0, \\ \psi_t = u_1\psi_x - \frac{1}{2}u_{1x}\psi, \end{cases} \tag{17}$$

for the KdV equation

$$u_{1t} = 3u_1u_{1x} - u_{1xxx}. \tag{18}$$

Example 2. Harry-Dym (HD) equation.

For the HD equation, the Schwarzian form reads

$$F_{HD}(p_i) = p_1^2 - \frac{2}{p_2} = 0, \tag{19}$$

which leads to the relation

$$v = -\frac{1}{u_1^2}. \tag{20}$$

From (14) and (20) we get the known Lax pair

$$\begin{cases} \psi_{xx} - \frac{1}{u_1^2}\psi = 0, \\ \psi_t = u_1\psi_x - \frac{1}{2}u_{1x}\psi \end{cases} \tag{21}$$

for the HD equation

$$u_{1t} = \frac{1}{4}u_1^3u_{1xxx}. \tag{22}$$

Example 3. Modified Boussinesq equation and Boussinesq equation.

The Schwarzian form of the modified Boussinesq (MBQ) equation (and the Boussinesq equation) is

$$F_{\text{MBQ}}(p_i) = p_{2x} + 3p_1p_{1x} + 3p_{1t} = 0. \tag{23}$$

Using (23) and (6), we have

$$v = \frac{3}{4}u_1^2 + \frac{3}{2}\int u_{1t}\mathrm{d}x. \tag{24}$$

Substituting (24) into (14) yields a Lax pair

$$\begin{cases} \psi_{xx} - \Big(\frac{3}{4}u_1^2 + \frac{3}{2}\int u_{1t}\mathrm{d}x\Big)\psi = 0, \\ \psi_t = u_1\psi_x - \frac{1}{2}u_{1x}\psi. \end{cases} \tag{25}$$

The related compatibility condition of (25) leads to

$$3u_1^2u_{1x} + 3u_{1x}\int u_{1t}\mathrm{d}x - \frac{1}{2}u_{1xxx} - \frac{3}{2}\int u_{1tt}\mathrm{d}x = 0. \tag{26}$$

Equation (26) is called the modified Boussinesq equation because it is linked with the known Boussinesq equation

$$u_{tt} + \Big(3u^2 + \frac{1}{3}u_{xx}\Big)_{xx} = 0 \tag{27}$$

by the Miura transformation

$$u = \frac{1}{3}(\pm u_{1x} - u_1^2 - \int u_{1t}\mathrm{dx}). \tag{28}$$

Example 4. Generalized fifth order KdV (FOKdV) equation.

The generalized fifth order Schwartzian KdV equation has the form

$$F_{\mathrm{FOKdV}}(p_i) = p_1 - a_1 p_{2xx} - a_2 p_2^2 = 0, \tag{29}$$

where a_1 and a_2 are arbitrary constants. From (29) and (6) we have

$$u_1 = -2a_1 v_{xx} + 4a_2 v^2. \tag{30}$$

Substituting (30) into (14), we get

$$\begin{cases} \psi_{xx} - v\psi = 0, \\ \psi_t = (a_1 v_{xxx} - 4a_2 v v_x)\psi \\ \qquad - 2(a_1 v_{xx} - 2a_2 v^2)\psi_x. \end{cases} \tag{31}$$

The related compatibility condition of (31) generates the generalized FOKdV equation

$$v_t - a_1 v_{xxxxx} + 4(a_1 + a_2) v v_{xxx} \tag{32}$$
$$+ 2(a_1 + 6a_2) v_x v_{xx} - 20 a_2 v^2 v_x = 0.$$

Some well-known fifth order integrable partial differential equations listed in [2] are just the special cases of (32). The usual FOKdV equation is related to (32) when

$$a_1 = 1,\ a_2 = \frac{3}{2}. \tag{33}$$

The Caudry-Dodd-Gibbon-Sawada-Kortera equation is related to (32) if

$$a_1 = 1,\ a_2 = \frac{1}{4}, \tag{34}$$

while the parameters a_1 and a_2 for the Kaup-Kupershmidt equation read

$$a_1 = 1,\ a_2 = 4. \tag{35}$$

Example 5. Generalized seventh order KdV (SOKdV) equation.

The generalized seventh order Schwartzian KdV equation is

$$F_{\mathrm{SOKdV}}(p_i) = p_1 - p_{2xxxx} - \alpha p_2 p_{2xx} - \beta p_{2x}^2 - \lambda p_2^3 = 0, \tag{36}$$

where α, β and λ are arbitrary constants. Using (36) and (6), we obtain

$$u_1 = -2v_{xxxx} + 4\alpha v v_{xx} + 4\beta v_x^2 - 8\lambda v^3. \tag{37}$$

Substituting (37) into (14), we get

$$\begin{cases} \psi_{xx} - v\psi = 0, \\ \psi_t = (v_{xxxxx} - 2(\alpha + 2\beta) v_x v_{xx} \\ \qquad - 2\alpha v v_{xxx} + 12\lambda v^2 v_x)\psi \\ \qquad + (-2v_{xxxx} + 4\alpha v v_{xx} + 4\beta v_x^2 - 8\lambda v^3)\psi_x. \end{cases} \tag{38}$$

The related compatibility condition of (38) gives the generalized SOKdV equation

$$v_t - v_{xxxxxxx} + 2(\alpha + 2) v v_{xxxxx} \tag{39}$$
$$+ 2(1 + 2\beta + 3\alpha) v_x v_{xxxx} - (16\beta + 12\alpha + 72\lambda) v v_x v_{xx}$$
$$+ (8\alpha v_{xx} - 8\alpha v^2 + 4\beta v_{xx} - 12\lambda v^2) v_{xxx}$$
$$- (24\lambda + 4\beta) v_x^3 + 56\lambda v^3 v_x = 0.$$

The usual SOKdV equation is related to (39) for

$$\alpha = 5,\ \beta = \frac{5}{2},\ \lambda = \frac{5}{2}. \tag{40}$$

The seventh order CDGSK equation corresponds to (39) if

$$\alpha = 12,\ \beta = 6,\ \lambda = \frac{32}{3}. \tag{41}$$

The seventh order KK equation is obtained if

$$\alpha = \frac{3}{2},\ \beta = \frac{3}{4},\ \lambda = \frac{1}{6}. \tag{42}$$

Example 6. Riccati equation (RE).

If the Schwarzian form (3) is taken as

$$F_{\mathrm{RE}}(p_i) \equiv p_2 p_1^2 + p_1 p_{1xx} + p_{1xt} \tag{43}$$
$$- \frac{1}{2} p_{1x}^2 - p_{1x} p_{1t} p_1^{-1} = 0,$$

then we have

$$v = \frac{1}{2}u_1^{-1}u_{1xx} + \frac{1}{4}u_1^{-2}(2u_{1xt} - u_{1x}^2) - \frac{1}{2}u_1^{-3}u_{1t}u_{1x}. \tag{44}$$

Consequently, the evolution equation of u_1 reads

$$3u_1u_{1t}u_{1xt} - u_1^2u_{1xtt} + u_1u_{1x}u_{1tt} - 3u_{1x}u_{1t}^2 = 0, \tag{45}$$

while the related Lax pair is

$$\begin{cases} \psi_{xx} - \Big(\frac{1}{2}u_1^{-1}u_{1xx} + \frac{1}{4}u_1^{-2}(2u_{1xt} - u_{1x}^2) - \frac{1}{2}u_1^{-3}u_{1t}u_{1x}\Big)\psi = 0, \\ \psi_t = \Big(-\frac{1}{2}u_{1x} + \lambda_1\Big)\psi + u_1\psi_x. \end{cases} \tag{46}$$

Actually (45) is equivalent to a trivial linearizable Riccati equation

$$w_t = w^2 + f_1(x) \tag{47}$$

under the transformation

$$u_1 = \exp\Big(2\int w\,\mathrm{d}t\Big), \tag{48}$$

where $f_1(x)$ is an arbitrary function of x. It is worth emphasizing again that the well known (1+1)-dimensional ShG model and MDB model are just the non-invertible Miura type deformation of the Riccati equation [4].

Except for the general form of the FOKdV equation, the SOKdV equations and the expression (43), all the other special examples mentioned in this section can be found in [2].

4. Special Extensions in Higher Dimensions

From Sect. 2, we know that the key procedure to obtain a Lax pair from the general conformal invariant form (3) is to find a suitable Lax form ansatz (like (4)) and a suitable relation ansatz (like (5)) between the field of the Schwarzian form and the spectral function such that the conformal invariants (p_i) becomes spectral function independent variables (P_i) (see (8)).

To extend this idea to higher dimensions is rather difficult. We hope to solve this problem in our future studies. In this section we offer some special extensions with the same Lax pair form (4).

If all the fields are functions of not only $\{x,\ t\}$ but also $\{y,\ z,\ \ldots,\}$, then all the formal theory is still valid as long as the independent conformal invariants in (6) is still restricted as p_1 and p_2, while the functional F of (6) also includes some derivatives and/or integrations of p_1 and p_2 with respect to other space variables $y,\ z,\ \ldots$ etc. Here we list only two special examples:

Example 7. (2+1)-dimensional KdV type breaking soliton equation.

The concept of breaking soliton equations has been developed in [9] and [10] by extending the usual constant spectral problem to a non-constant spectral problem. Various interesting properties of the breaking soliton equations have been revealed by many authors. For instance, infinitely many symmetries of some breaking soliton equations are given in [11, 12]. In [13], it has been pointed out that every (1+1)-dimensional integrable model can be extended to some higher dimensional breaking soliton equations by means of its strong symmetries. Yu and Toda [14] has derived the Schwarzian form of the (2+1)-dimensional KdV type breaking soliton equation which reads

$$F_{\text{2dSKdV}} \equiv p_1 + \int p_{2y}\,\mathrm{d}\,x = 0. \tag{49}$$

Then from (6) and (49), we have

$$u_1 = 2\int v_y\,\mathrm{d}\,x. \tag{50}$$

Substituting (49) into (14), we obtain a Lax pair

$$\begin{cases} \psi_{xx} = v\psi, \\ \psi_t = 2\int v_y \mathrm{d}x\psi_x - (v_y - \lambda_1)\psi, \end{cases} \tag{51}$$

for the (2+1)-dimensional KdV type breaking soliton equation

$$v_t = -v_{xxy} + 4vv_y + 2v_x\int v_y\,\mathrm{d}\,x \equiv \Phi v_y, \tag{52}$$

where Φ is just the strong symmetry of the (1+1)-dimensional KdV equation.

Example 8. (2+1)-dimensional fifth order equation.

If making the replacement

$$p_2 \to \int p_{2y} \mathrm{d}\, x \tag{53}$$

to some examples listed in the last section, we can obtain some special types of their (2+1)-dimensional extensions. Example 7 is rightly obtained from the (1+1)-dimensional KdV equation by using this replacement.

A generalization of the fifth order Schwarzian equation (29) reads

$$p_1 = b_1 p_{2xy} + b_2 p_{xx} + c_1 \Big(\int p_{2y} \mathrm{d}\, x \Big)^2 + c_2 p_2 \int p_{2y} \mathrm{d}\, x + c_3 p_2^2, \tag{54}$$

where $b_1,\ b_2,\ c_1,\ c_2$ and c_3 are arbitrary constants. From (6) and (54) we know that

$$u_1 = -2b_1 v_{xy} - 2b_2 v_{xx} + 4c_1 \Big(\int v_y \mathrm{dx} \Big)^2 + 4c_2 v \Big(\int v_y \mathrm{dx} \Big) + 4c_3 v^2, \tag{55}$$

and the related Lax pair becomes

$$\begin{cases} \psi_{xx} = v\psi, \\ \psi_t = \Big(4c_1 \big(\int v_y \mathrm{d}\, x \big)^2 + 4c_2 v \big(\int v_y \mathrm{d}\, x \big) + 4c_3 v^2 \\ \qquad - 2b_1 v_{xy} - 2b_2 v_{xx} \Big) \psi_x \\ \quad + \Big(\lambda_1 - 2v_x \big(2c_3 v + c_2 \int v_y \mathrm{d}\, x \big) \\ \quad - 2v_y \big(2c_1 \int v_y \mathrm{d}\, x + c_2 v \big) + b_1 v_{xxy} + b_2 v_{xxx} \Big) \psi. \end{cases}$$

The corresponding evolution equation for the field v is

$$\begin{aligned} v_t = {} & b_1 v_{xxxxy} + b_2 v_{xxxxx} \\ & + 2v_x (4c_2 v^2 - 6c_1 v_{xy} - 3c_2 v_{xx} + 8c_1 v \textstyle\int v_y \mathrm{d}\, x) \\ & - 2(c_2 v + 2b_1 v + 2c_1 \textstyle\int v_y \mathrm{d}\, x) v_{xxy} \\ & - 2(2c_3 v + 2b_2 v + c_2 \textstyle\int v_y \mathrm{d}\, x) v_{xxx} \\ & + 2v_x \Big(10c_3 v^2 + 2c_1 (\textstyle\int v_y \mathrm{d}\, x)^2 + 6c_2 v \textstyle\int v_y \mathrm{d}\, x \\ & - (b_2 + 6c_3) v_{xx} - (b_1 + 3c_2) v_{xy} \Big). \end{aligned} \tag{56}$$

It is obvious that when $y = x$ and/or $v_y = 0$, the (2+1)-dimensional fifth order equation (56) will be reduced back to the (1+1)-dimensional FOKdV equation (32) for $b_1 + b_2 = a_1$ and $c_1 + c_2 + c_3 = a_2$.

5. On Spectral Parameters

In the last two sections, we have omitted the spectral parameter(s). In order to introduce the possible spectral parameter(s) into the Lax pairs, one can use the symmetry transformations of the original nonlinear models. In some cases, to find a symmetry transformation such that a nontrivial parameter can be included in the Lax pair (4) is quite easy. For instance, it is well known that the KdV equation (18) is invariant under the Galilei transformation

$$u_1 \to u_1(x + 3\lambda t,\ t) + \lambda \equiv u_1(x',\ t) + \lambda. \tag{57}$$

In (57) we use the parameter $x+3\lambda t = x'$ for simplicity in other formulas especially in the Lax pair (see (58)). Substituting (57) into (17) yields the usual Lax pair of the KdV equation with the spectral parameter λ:

$$\begin{cases} \psi_{xx} - \dfrac{1}{2}(u_1 + \lambda)\psi = 0, \\ \psi_t = (u_1 - 2\lambda)\psi_x - \dfrac{1}{2} u_{1x} \psi, \end{cases} \tag{58}$$

where x' has been rewritten as x.

However, for some other models, to add the parameters to (4) can not be easily realized. In those cases, the spectral parameters have to be included in (4) in very complicated ways. For example, as for the CDGSK equation ((32) with (34)), we failed to include a nontrivial spectral parameter by using its point Lie symmetries. Nevertheless, if we use the higher order symmetries and/or nonlocal symmetries of the model we can include some nontrivial parameters in (31) with (34). For instance, still for the CDGSK equation, if ψ_1 is a special solution of (31) with (34), one can prove that

$$u' = u - 6 \frac{\lambda(\lambda\psi_1^2 - \lambda\psi_{1x} p - 6\psi_{1x})}{(\lambda p + 6)^2}, \tag{59}$$

with

$$p_x = \psi_1 \tag{60}$$

is also a solution of the CDGSK equation. By substituting (59) into (31), we obtain a second order Lax pair ($P = 6 + \lambda p$)

$$\begin{cases} \psi_{xx} = -\Big(u - 6\dfrac{\lambda(\lambda\psi_1^2 - \lambda\psi_{1x}p - 6\psi_{1x})}{(\lambda p + 6)^2}\Big)\psi, \\ \psi_t = \Big(\dfrac{6\lambda(u_x\psi_1)_x}{P} - 12\dfrac{\psi_1\lambda^2(3u\psi_{1x} + 2u_x\psi_1)}{P^2} \\ \qquad + 72\lambda^3\psi_1\dfrac{u\psi_1^2 - \psi_{1x}^2}{P^3} - u_{xxx} - ww_x \\ \qquad\qquad - 36\lambda^4\psi_1^3\dfrac{2\lambda\psi_1^2 - 5\psi_{1x}P}{P^5}\Big)\psi \\ \qquad + \Big(36\lambda^2 u\dfrac{\psi_1^2}{P^2} + 2u_{xx} + u^2 - 12\lambda\psi_{1x}\dfrac{u_x}{P} \\ \qquad\qquad - 36\lambda^4\dfrac{\psi_1^4}{P^4} + 72\lambda^3\psi_1^2\dfrac{\psi_{1x}}{P^3}\Big)\psi_x \end{cases} \quad (61)$$

for the CDGSK model with ψ_1 being a solution of (31). Obviously, the Lax pair (61) is too complicated for real applications. In order to eliminate ψ_1 from (61), one has to heighten the order of the Lax pair. Actually, the simplest Lax pairs for the CDGSK equation, the KK equation and the Boussinesq equation are of order three [15]. Though the second order Lax pairs may be useless to get exact solutions by inverse scattering transformation for the fifth order (and seventh order) CDGSK and KK equations, they are still useful to find some other types of interesting properties like the nonlocal symmetries [16] and the exact solutions related to the nonlocal symmetries [17].

In many other cases, one can not introduce nontrivial spectral parameter(s) to the Lax pair (14) at all. The (1+1)-dimensional Lax pairs without nontrivial spectral parameter(s) are called fake Lax pairs. In [18], Calogero and Nucci have pointed out that any (1+1)-dimensional nonlinear equations with one conservation law possess fake Lax pairs. For the fake Lax pair(s), the trivial parameters can be eliminated by proper gauge [19].

6. Summary and Discussions

In summary, every (1+1)-dimensional equation which has a Schwarzian form seems to possess a second order Lax pair. In this paper we have proven this conclusion when the Schwarzian form is an arbitrary functional of two conformal invariants and their derivatives and integrates.

Usually, the Lax operators for various integrable models (except for the KdV hierarchy) are taken as higher order operators. Because the order of the Lax pair operators for some models has been lowered, the spectral parameter has disappeared. In order to recover some types of nontrivial spectral parameters, we have to use the symmetries of the original nonlinear equations, and then the spectral parameter(s) will appear in the second order Lax operator in some complicated ways. As is known, many interesting properties of some special models from the Lax pairs without spectral parameters have been successfully obtained [20, 21]. For instance, in [20, 16], infinitely many nonlocal symmetries of the KdV equation, HD equation, CDGSK equation and the KK equation have been derived from the spectral parameter independent Lax pairs. Therefore, how to obtain some other integrable properties from the Lax pairs listed here for general or special models is worthy of further study though the spectral parameters are not included explicitly in them.

In addition, The conclusion for the general (1+1)-dimensional Schwarzian equations can also be extended to some special types of (2+1)-dimensional models, like the breaking soliton equations. However, how to extend the method and the conclusions to general (2+1)-dimensions or to even higher dimensions is still open.

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